Comparison of Model Order Reduction Methods like POD, CVT, Arnoldi-Krylov and PGD to solve quasistatic problems

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In the domain of numerical computation, Model Order Reduction (MOR) methods are more and more applied in mechanics and have shown their efficiency in terms of computation time and memory requirements. In computational electromagnetics, research has started recently and the different methods available in the literature need to be compared in order to find the most efficient one. We propose to evaluate MOR approaches in order to solve linear magnetoquasistatic field problems. Therefore, the Proper Orthogonal Decomposition (POD), the Centroidal Voronoi Tessellation (CVT), the Proper Generalized Decomposition (PGD) and the Arnoldi-Krylov projection (AKP) are developed and compared.

***Index Terms*—Arnoldi-Krylov projection, Centroidal Voronoi Tessellation, Model Order Reduction, Proper Generalized Decomposition, Proper Orthogonal Decomposition, Quasistatic field**

I. Introduction

MODELING electromagnetic devices with the Finite Element Method (FEM) associated with a time-stepping scheme has proved to produce very accurate results over the past few years. This approach requires solving large-scale systems on a time interval which induces a prohibitive computational time. To circumvent this issue, model order reduction methods have been proposed in the literature. These methods consist in performing a projection of the solution of the full problem onto a reduced basis. Then, the large-scale system is dramatically reduced leading to a significant calculation speedup. In numerical computation, several methods have been presented to generate this basis. These approaches have been developed in order to solve engineering problems like solid and fluid mechanics or electric networks. Recently, some methods available in the literature have been applied in the case of electromagnetics [1] [2] [3].

In this communication, we propose to develop and to compare several MOR methods with a potential formulation used to solve quasistatic vector potential formulation. Therefore, the studied approaches are the Proper Orthogonal Decomposition (POD) [4] [5], the Centroidal Voronoi Tessellation (CVT) [6], the Arnoldi-Krylov projection (AKP) [7] and the Proper Generalized Decomposition (PGD) [8]. First, the numerical model obtained from the modified vector potential formulation is briefly presented and the MOR approaches are developed. Then, an academic example is solved using the different MOR methods which are compared in terms of accuracy with the full Finite Element Model.

II. Model Order Reduction Methods

Let us consider a magnetoquasistatic problem defined on a domain D containing a conducting domain D_c. For sake of clarity, we will assume that D contains only one stranded inductor, even though the following approaches remain valid with several stranded inductors. Applying the Finite Element Method to a modified magnetic vector potential formulation leads to this general system of algebro-differential equation:

$$\boldsymbol{N}\frac{\mathrm{d}\boldsymbol{X}(t)}{\mathrm{d}t}+\boldsymbol{M}\boldsymbol{X}(t)=\boldsymbol{F}i(t) \tag{1}$$

where $\boldsymbol{X}(t)$ denotes the vector solution of size n, $\boldsymbol{M}$ and $\boldsymbol{N}$ are square matrices depending on the magnetic permeability and electric conductivity respectively. $\boldsymbol{F}$ is the source term depending on the inductor shape and $i(t)$ its current. From this equation, a MOR approach may be achieved by finding a well-adapted reduced basis in which the following approximation holds: $\boldsymbol{X}(t)=\boldsymbol{\Psi}\boldsymbol{X}_r(t)$, where $\boldsymbol{\Psi}\in R^{n\times m},\ m<<n$ is a reduced basis. The vector $\boldsymbol{X}_r(t)$ denotes the reduced solution of size m. Introducing this decomposition in the equation (1) and projecting it into the reduced basis, according to the Ritz-Galerkin procedure lead to the reduced system of size m:

$$\boldsymbol{N}_r\frac{\mathrm{d}\boldsymbol{X}_r(t)}{\mathrm{d}t}+\boldsymbol{M}_r\boldsymbol{X}_r(t)=\boldsymbol{F}_r i(t) \tag{2}$$

where $\boldsymbol{N}_r=\boldsymbol{\Psi}^t\boldsymbol{N}\boldsymbol{\Psi}$, $\boldsymbol{M}_r=\boldsymbol{\Psi}^t\boldsymbol{M}\boldsymbol{\Psi}$ and $\boldsymbol{F}_r=\boldsymbol{\Psi}^t\boldsymbol{F}$. The key of MOR methods is to find the "best" reduced basis such that the approximation $\boldsymbol{X}(t)=\boldsymbol{\Psi}\boldsymbol{X}_r(t)$ of the solution $\boldsymbol{X}(t)$ of (1) introduces an error as small as possible.

A. Proper Orthogonal Decomposition (POD)

The POD approach [4] is generally based on the snapshots method [5] in order to generate a reduced basis efficiently. It requires solving the full system (1) at k different time steps. The k solutions $\boldsymbol{X}(t_j),\ j=1\dots k$, are concatenated in the so-called snapshots matrix $\boldsymbol{K}_s\in\mathbb{R}^{n\times k}$. A Singular Value Decomposition is then applied on this matrix: $\boldsymbol{K}_s=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}$, where $\boldsymbol{U}\in\mathbb{R}^{n\times n}$ and $\boldsymbol{V}\in\mathbb{R}^{k\times k}$ are unitary matrices. $\boldsymbol{\Sigma}\in\mathbb{R}^{n\times k}$ is zero except on its main diagonal. The reduced basis obtained from POD is finally $\boldsymbol{\Psi}=\boldsymbol{U}\boldsymbol{\Sigma}_r$, where $\boldsymbol{\Sigma}_r$ contains the m largest values of $\boldsymbol{\Sigma}$, $m\leq k$.

B. Centroidal Voronoi Tessellation (CVT)

The CVT approach is also based on the snapshots method [6]. It is a method of partitioning the snapshots in $\mathbb{R}^n$. Let us consider k snapshots $\boldsymbol{X}(t_j),\ j = 1 \dots k$, concatenated into the snapshot matrix $\boldsymbol{K}_s \in \mathbb{R}^{n \times k}$. The CVT method consists in finding m clusters of snapshots $V_i,\ i = 1 \dots m,\ m \leq k$. Each cluster has a so-called seed $\boldsymbol{\Psi}_i \in \mathbb{R}^n$, which has the following property:

$$V_i = \{\boldsymbol{X}(t_j) / \left\|\boldsymbol{X}(t_j) - \boldsymbol{\Psi}_i\right\| < \left\|\boldsymbol{X}(t_j) - \boldsymbol{\Psi}_j\right\|,\ j \neq i\} \quad (3)$$

Each seed is the center of mass of its cluster. The Lloyd's algorithm allows to construct this CVT [6]. The reduced basis $\boldsymbol{\Psi}$ is the concatenation of the m seeds $\boldsymbol{\Psi}_i, i = 1 \dots m$.

C. Arnoldi-Krylov Projection (AKP)

The AKP method is based on the Pade's approximation of the transfer function [7]. The Laplace transform of problem (1) is $\boldsymbol{h}(s) = (s\boldsymbol{N} + \boldsymbol{M})^{-1}\boldsymbol{F}$, where s is the Laplace variable. Performing a power series expansion and the Pade approximation centered at ζ on $\boldsymbol{h}(s)$ leads to:

$$\boldsymbol{h}(s) = \sum_{i=0}^{\infty} \boldsymbol{h}_j (s - \zeta)^j \quad (4)$$

$$\boldsymbol{h}_j = \left[-(\zeta\boldsymbol{N} + \boldsymbol{M})^{-1}\boldsymbol{M}\right]^j (\zeta\boldsymbol{N} + \boldsymbol{M})^{-1}\boldsymbol{F} \quad (5)$$

with ζ the expansion point, which is an angular frequency. The AKP method consists in truncating the infinite sum in $\boldsymbol{h}(s)$ to m terms, denoted by $\tilde{\boldsymbol{h}}(s)$ such that $\tilde{\boldsymbol{h}}(s) \in \mathcal{K}_m = \mathrm{Vect}(\boldsymbol{h}_0, \boldsymbol{h}_1, \dots, \boldsymbol{h}_{m-1})$. Thus, constructing an orthonormal basis $\boldsymbol{\Psi} \in \mathbb{R}^{n \times m}$ on $\mathcal{K}_m$ appears to be a good candidate for the reduced basis. By choosing ζ as a real angular frequency, the m vectors $\boldsymbol{h}_j,\ j = 0 \dots m-1$ are real. Therefore, this approach remains valid in the time domain.

D. Proper Generalized Decomposition (PGD)

The PGD method is based on a separated representation of functions of the solution such that $\boldsymbol{X}_m(t) = \sum_{i=1}^{m} \boldsymbol{\Psi}_i X_{r,i}(t)$ [8]. With this method, the functions $\boldsymbol{\Psi}_i$ are not known. To define the functions $\boldsymbol{\Psi}_i$ and $X_{r,i}(t)$, an iterative procedure is used. At the k^{th} iteration, the solution is sought as $\boldsymbol{X}_k(t) = \sum_{i=1}^{k} \boldsymbol{\Psi}_i X_{r,i}(t) = \boldsymbol{X}_{k-1}(t) + \boldsymbol{\Psi}_k X_{r,k}(t)$ where only $\boldsymbol{\Psi}_k$ and $X_{r,k}(t)$ are the unknowns of the problem. By injecting this into (1), a residual function R may be defined as:

$$\mathrm{R}(\boldsymbol{\Psi}_k, X_{r,k}(t)) = \boldsymbol{N}\frac{\mathrm{d}\boldsymbol{X}_k(t)}{\mathrm{d}t} + \boldsymbol{M}\boldsymbol{X}_k(t) - \boldsymbol{F}i(t) \quad (6)$$

A spatial operator S_k and a time operator T_k may be defined as:

$$S_k \ : \ \boldsymbol{\Psi}_k \ \mapsto \ (\mathrm{R}(\boldsymbol{\Psi}_k, X_{r,k}(t)) \cdot X_{r,k}(t)) \quad (7)$$

$$T_k \ : \ X_{r,k} \ \mapsto \ (\mathrm{R}(\boldsymbol{\Psi}_k, X_{r,k}(t)) \cdot \boldsymbol{\Psi}_k) \quad (8)$$

Thus, solving $S_k(\boldsymbol{\Psi}_k) = 0$ and $T_k(X_{r,k}) = 0$ with a fixed point method leads to identifying both $\boldsymbol{\Psi}_k$ and $X_{r,k}(t)$.

III. Application

A 2D linear magnetodynamic problem composed of an iron core, a conducting plate and an inductor supplied with a 1kHz sinusoidal current is studied. The backward Euler method is used to solve (1) and (2) on five periods with a 25μs time step. The 2D mesh is made of 1505 nodes and 2937 triangles. The Joule losses $\mathcal{P}$ obtained from the MOR methods are compared with those given by the full problem (1). An error estimator is defined by $\mathcal{E}(\mathcal{P}) = \frac{\|\mathcal{P}_{ref} - \mathcal{P}_{red}\|}{\|\mathcal{P}_{ref}\|}$. Figure 2 presents the evolution of $\mathcal{E}(\mathcal{P})$ versus the number of large-scale computations carried out by the methods. Those computations result from solving (5) and (7) for AKP and PGD, and (1) for POD and CVT. For a given error equals to 10^{-6}, the speedup factor is 7.6, 11.5, 12.7 and 3.8 for the POD, CVT, AKP and PGD approaches respectively (snapshots computation included). On this example, the PGD approach appears to be very precise, but it is also the slowest method. Both POD and CVT techniques give the same order of results. The AKP method offer the best precision and is also the fastest.

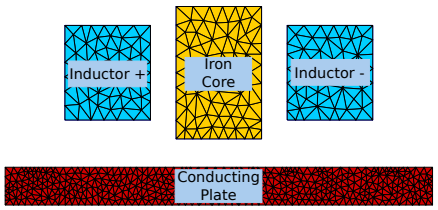

Fig. 1. 2D mesh of the problem

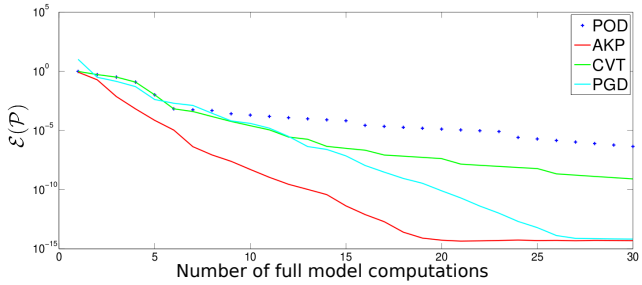

Fig. 2. Relative Joule losses error in the conductor versus the number of large-scale computations (semi-log scale)

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